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REDUCING HEEGAARD SPLITTINGS

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If a Heegaard splitting of a nonsufficiently large 3-manifold has the property that there exist essential disks, one in each of the two Heegaard handlebodies, whose boundaries are disjoint, then the splitting is reducible.

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nonsufficiently large 3-manifold
reducible Heegaard splitting

0. Introduction

Haken has shown that if a closed 3-manifold with a given Heegaard splitting contains an essential 2-sphere, then it contains one which meets the Heegaard surface in a single circle [8]. We observe that Haken's argument extends to spheres and disks in Heegaard splittings of manifolds with boundary, and give some applications of this observation. (The fact that Haken's argument can be extended in this way has also been noted by Bonahon and Otal [3].)

The main application (given in Section 3) is to prove the result announced in [4], that if a Heegaard splitting of a closed 3-manifold has the property that there exist essential disks, one in each of the two Heegaard handlebodies, whose boundaries are disjoint, then either the splitting is reducible or the manifold contains an incompressible surface. This result seems potentially useful in dealing with Heegaard splittings of non-Haken manifolds, for instance homotopy 3-spheres (see Corollary 3.2). We have learned from P. Shalen that he was also aware of the result in this case.

As another application, we give a sufficient condition for the addition of 2-handles to a 3-manifold to produce an irreducible manifold with incompressible boundary. In the case of a single 2-handle, this gives an alternative proof of a theorem of Jaco [11]. This is the content of Section 2.

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Finally, in Section 4 we answer a question of Maskit (see [15]) by showing that if W is a handlebody and the homotopy class of a simple loop in ∂W is a proper power in $\pi_1(W)$, then it is a proper power of an element which is represented by a simple loop in ∂W. This has also been proved by P. Scott (unpublished). Actually we prove the natural extension of this to compression bodies.

In Section 1 we give some preliminary definitions and state the extension of Haken's lemma, Lemma 1.1. We also observe that it implies additivity of Heegaard genus under boundary connected sum.

1. Haken's lemma

All manifolds will be compact and orientable.

A 2-sphere in a 3-manifold M is *essential* if it does not bound a 3-ball in M. A 3-manifold M is *irreducible* if it contains no essential 2-sphere.

Let F be a surface in a 3-manifold M which is either properly embedded or contained in ∂M. An *essential disk* in (M, F) is a disk D in M such that $D \cap F = \partial D$ and ∂D is essential in F. If such a disk exists, F is *compressible* in M; otherwise, it is *incompressible.*

If F is contained in ∂M, the pair (M, F) is *irreducible* if M is irreducible and F is incompressible in M. Thus, if $F \neq \emptyset$, (M, F) is irreducible if and only if every disk $(D, \partial D) \subset (M, F)$ is isotopic rel ∂ to a disk in F.

A *compression body* W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times I \cup 2\text{-handles} \cup 3\text{-handles}$ and $\partial_- W$ has no 2-sphere components. It follows that $(W, \partial_- W)$ is irreducible. If $\partial_+ W$ is closed and connected and $\partial_- W = \emptyset$, W is a *handlebody.* If $W \cong \partial_+ W \times I$, W is *trivial.*

A *complete disk system* $\boldsymbol{D}$ for a connected compression body W is a disjoint union of disks $(\boldsymbol{D}, \partial \boldsymbol{D}) \subset (W, \partial_+ W)$ such that W cut along $\boldsymbol{D}$ is homeomorphic to

$$\begin{cases} \partial_- W \times I, & \text{if } \partial_- W \neq \emptyset, \\ B^3, & \text{if } \partial_- W = \emptyset. \end{cases}$$

In general, a complete disk system for W is a union of complete disk systems for the components of W. Note that for any handle decomposition of W as above, the union of the cores of the 2-handles (extended by collars on their boundaries to $\partial_+ W$) contains a complete disk system for W.

If F is a surface and α is a closed 1-manifold in F, then $W(F; \alpha)$ will denote the compression body $F \times I \cup 2\text{-handles} \cup 3\text{-handles}$ where the attaching circles of the 2-handles are the components of α.

A 3-*manifold triad* $(M; B, B')$ is a cobordism M rel ∂ between surfaces B and B'. Thus B and B' are disjoint surfaces in ∂M with $\partial B \cong \partial B'$, such that $\partial M = B \cup B' \cup \partial B \times I$.

A *Heegaard splitting* of $(M; B, B')$ is a pair (W, W') where W, W' are compression bodies such that $W \cup W' = M$, $W \cap W' = \partial_+ W = \partial_+ W' = F$, say, and $\partial_- W = B$,

$\partial_- W' = B'$. Thus $\partial F \cong \partial B \cong \partial B'$. Any triad $(M; B, B')$ such that $B \cup B'$ has no 2-sphere components has a Heegaard splitting.

Let F be a surface and α a closed 1-manifold in F. We denote by $\sigma(F; \alpha)$ the surface obtained from F by doing 1-surgeries along the components of α. In particular, note that $\partial_- W(F; \alpha)$ is homeomorphic to $\sigma(F; \alpha)$ with all 2-sphere components removed.

Let F be a surface in a 3-manifold M, and let D be a disjoint union of disks in M such that $D \cap F = \partial D$. We may then do *ambient* 1*-surgery on F along D* to obtain a surface in M homeomorphism to $\sigma(F; \partial D)$.

The following lemma is a mild generalization of the main result of [8].

Lemma 1.1. *Let (W, W') be a Heegaard splitting of $(M; B, B')$. Let $(S, \partial S) \subset (M, B \amalg B')$ be a disjoint union of essential 2-spheres and disks. Then there exists a disjoint union of essential 2-spheres and disks S^* in M such that*

(i) *S^* is obtained from S by ambient 1-surgery and isotopy;*
(ii) *each component of S^* meets F in a single circle;*
(iii) *there exist complete disk systems $\boldsymbol{D}$, $\boldsymbol{D}'$ for W, W' respectively such that $\boldsymbol{D} \cap S^* = \boldsymbol{D}' \cap S^* = \emptyset$.*

Note that if M is irreducible (in which case S must consist of disks) then it follows that S^* is isotopic to S.

A similar generalization of Haken's result has been noted independently by Bonahon and Otal. As their proof has since appeared in print [3], we will not include a proof of Lemma 1.1 here, but simply refer the reader to [8] (see also Jaco's account of Haken's proof [10, Chapter II]) and [3].

If M is a 3-manifold, let $\hat{M}$ denote the manifold obtained by capping off all 2-sphere boundary components of M with 3-balls. Then, following Waldhausen [19], define the *Heegaard genus* $g(M)$ of M to be $\min\{\beta_1(W): (W, W')$ is a Heegaard splitting of $(\hat{M}; \emptyset, \partial\hat{M})\}$. Note that, here, W is a disjoint union of handlebodies. We refer to $\beta_1(W)$ as the *genus* of the Heegaard splitting (W, W').

Haken's original lemma, together with Milnor's uniqueness theorem for prime connected sum decompositions [16], implies that for closed 3-manifolds, Heegaard genus is additive under connected sum. Since uniqueness extends to connected sum decompositions of manifolds with boundary (see [9, Theorem 3.21]), Lemma 1.1 (in the case of spheres) shows that genus is additive under connected sum in general. We note that Lemma 1.1 for disks implies additivity under boundary connected sum.

Corollary 1.2. *Heegaard genus is additive under boundary connected sum.*

Proof. Suppose that $M = M_1 \#_\partial M_2$. Clearly $g(M) \leqslant g(M_1) + g(M_2)$. We shall show that $g(M) \geqslant g(M_1) + g(M_2)$. We assume without loss of generality that M is connected.

First assume that M is irreducible. Let D be the disk in M that realizes the decomposition $M = M_1 \#_\partial M_2$. If ∂D is inessential in ∂M, then M_2 (say) is a 3-ball and the result is trivial. So suppose that ∂D is essential in ∂M, and let (W, W') be a Heegaard splitting of $(M; \emptyset, \partial M)$ of genus $g(M)$. Since M is irreducible, Lemma 1.1 implies that D may be isotoped so that $D \cap W$ is a disk (which necessarily separates W into two handlebodies W_1 and W_2, say), and so that the annulus $D \cap W'$ separates W' into two compression bodies W_1' and W_2'. Then (W_i, W_i') is a Heegaard splitting of $(M_i; \emptyset, \partial M_i)$ of genus g_i, $i = 1, 2$, where $g_1 + g_2 = g(M)$. This shows that $g(M_1) + g(M_2) \leq g(M)$.

For the general case, suppose $M = M_1 \#_\partial M_2$, and let

$$M_1 = \mathop{\#}_{i=1}^{m} N_1^{(i)}, \qquad M_2 = \mathop{\#}_{j=1}^{n} N_2^{(j)}$$

be prime connected sum decompositions of M_1 and M_2. By renumbering the N's if necessary, we may assume that the boundary connected sum $M_1 \#_\partial M_2$ takes place over boundary components of $N_1^{(1)}$ and $N_2^{(1)}$. Thus

$$M = (N_1^{(1)} \#_\partial N_2^{(1)}) \# \mathop{\#}_{i=2}^{m} N_1^{(i)} \# \mathop{\#}_{j=2}^{n} N_2^{(j)}.$$

Since $N_1^{(1)}$ and $N_2^{(1)}$ are prime and have non-empty boundary, they are irreducible. Hence $N_1^{(1)} \#_\partial N_2^{(1)}$ is irreducible and so $g(N_1^{(1)} \#_\partial N_2^{(1)}) = g(N_1^{(1)}) + g(N_2^{(1)})$ by the previous paragraph. The fact that $g(M) = g(M_1) + g(M_2)$ now follows from the additivity of genus with respect to connected sum. $\square$

Remark. One can consider the operation of cutting a 3-manifold M along any (not necessarily separating) 2-sphere or disk. It is not hard to show that the statement that holds in this more general setting, and which generalizes the additivity of genus under connected sum and boundary connected sum, is that this operation leaves $g(M) - \beta_1(M)$ invariant.

2. Strongly irreducible Heegaard splittings

A Heegaard splitting (W, W') is *reducible* if there exist essential disks $(D, \partial D) \subset (W, F)$, $(D', \partial D') \subset (W', F)$ such that $\partial D = \partial D'$. Otherwise, it is *irreducible.*

Lemma 1.1 immediately implies that if a 3-manifold M has an irreducible Heegaard splitting then M is irreducible.

Let us say that (W, W') is *strongly irreducible* if neither W nor W' is trivial and there do not exist essential disks $(D, \partial D) \subset (W, F)$, $(D', \partial D') \subset (W', F)$ such that $\partial D \cap \partial D' = \emptyset$. Clearly, strong irreducibility implies irreducibility.

Then we have the following analogue of the above statement for pairs.

Theorem 2.1. *If $(M; B, B')$ has a strongly irreducible Heegaard splitting then $(M, B \amalg B')$ is irreducible.*

Proof. Let (W, W') be a Heegaard splitting of $(M; B, B')$ such that neither W nor W' is trivial, and suppose that $(M, B \amalg B')$ is reducible. Then there exists an essential 2-sphere or disk $(S, \partial S) \subset (M, B)$, say. By Lemma 1.1, we may assume that S meets F in a single circle and is disjoint from some complete disk system $\boldsymbol{D}$ for W. Let D be a component of $\boldsymbol{D}$ (which is necessarily non-empty). Then D and $S \cap W'$ are essential disks in W, W' respectively with disjoint boundaries, showing that (W, W') is not strongly irreducible. □

Let Q be an irreducible 3-manifold and F a surface in ∂Q. Let W be some compression body with $\partial_+ W = F$. Then $Q \bigcup_F W$ is obtained from Q by adding 2-handles along F (followed by 3-handles). Now suppose that F is compressible in Q, and let W' be the maximal compression body of F in Q (see [2, § 2]). If the Heegaard splitting (W, W') is strongly irreducible, then it follows easily from Theorem 2.1 that $(Q \bigcup_F W, \partial_- W)$ is irreducible.

In the special case that W has only a single 2-handle, it is easy to give a simple sufficient condition for (W, W') to be strongly irreducible. (A sufficient condition for strong irreducibility in general is described in [5].) We thus obtain the following result, which is essentially due to Jaco [11] (see also [6, 12, 18]). In the case where Q is a handlebody, it is due to Przytycki [17].

Corollary 2.2. *Let Q be an irreducible 3-manifold and let F be a surface in ∂Q which is compressible in Q. Let α be a simple loop in F such that $F - \alpha$ is incompressible in Q. Let W be the compression body $W(F; \alpha)$. Then $(Q \bigcup_F W, \partial_- W)$ is irreducible.*

Proof. Let W' be the maximal compression body for F in Q, and consider the Heegaard splitting (W, W'). We shall show that (W, W') is strongly irreducible; the irreducibility of $(Q \bigcup_F W, \partial_- W)$ then follows from Theorem 2.1.

First note that W' is non-trivial by hypothesis, and that W is also non-trivial since α is essential in F. Suppose that there exist essential disks $(D, \partial D) \subset (W, F)$, $(D', \partial D') \subset (W', F)$ such that $\partial D \cap \partial D' = \emptyset$. Let E be the core of the 2-handle in W, with $\partial E = \alpha$. Since E is a complete disk system for W, it is easy to show by a standard innermost circle—outermost arc argument that E may be isotoped so that $E \cap D = \emptyset$.

Since ∂D bounds a disk in W, it bounds a disk in $\sigma(F; \alpha)$. Therefore either ∂D is parallel to α on F, or ∂D bounds a punctured torus T_0 in F with α contained in T_0 and non-separating. In the first case, D' contradicts the incompressibility of $F - \alpha$ in Q. In the second case, either $\partial D' \subset F - T_0$, which again contradicts the incompressibility of $F - \alpha$, or $\partial D' \subset T_0$. But then we would have $\partial D \in \langle \partial D' \rangle$ in $\pi_1(F)$, so that ∂D would bound a disk in W', again contradicting the incompressibility of $F - \alpha$ in Q. □

3. Main theorem

While it is clear that strongly irreducible Heegaard splittings are irreducible, the converse implication is false in general (although it is not hard to see that it is true for splittings of genus 2). A simple example, suggested by the referee, is the genus 3 Heegaard splitting of the 3-torus $S^1 \times S^1 \times S^1$ obtained as follows: start with a regular neighborhood N of $S^1 \times S^1 \times \{\text{point}\}$, drill out a vertical tunnel from N, and add a 1-handle along the third S^1-factor. The resulting Heegaard splitting is necessarily irreducible, but clearly not strongly irreducible.

However, a partial converse can be obtained by restricting attention to closed 3-manifolds which are not sufficiently large.

Theorem 3.1. *Let (W, W') be a Heegaard splitting of a closed 3-manifold M. If (W, W') is not strongly irreducible then either (W, W') is reducible or M contains an incompressible surface of positive genus.*

Using the loop theorem—Dehn's lemma, we immediately obtain the following corollary. (This was previously known to P. Shalen (unpublished).)

Corollary 3.2. *The 3-dimensional Poincaré Conjecture is equivalent to the statement that for any Heegaard splitting (W, W') of positive genus of a homotopy 3-sphere, there exist essential loops α, α' in F which intersect in at most one point and which are null-homotopic in W, W' respectively.*

To prove Theorem 3.1 it is convenient to have a notion of complexity for a 1-submanifold of a closed surface. First define the complexity of a closed surface F by

$$\bar{c}(F) = \sum (1 - \chi(F_0)),$$

summed over all components F_0 of F that are not 2-spheres.

In particular, $\bar{c}(F) \geq 0$, with equality if and only if F is a disjoint union of 2-spheres.

Now if α is a 1-submanifold of F, define the complexity of α by

$$c(\alpha) = \bar{c}(F) - \bar{c}(\sigma(F; \alpha)).$$

Note that $c(\alpha) \geq 0$, with equality if and only if all components of α bound disks in F. More generally, if α and α' are disjoint 1-submanifolds of F, then $c(\alpha \cup \alpha') \geq c(\alpha)$, with equality if and only if $\langle \alpha' \rangle \subset \langle \alpha \rangle$ (normal closures in $\pi_1(F)$).

Proof of Theorem 3.1. By hypothesis there exist disks D, D' in W, W' respectively such that ∂D and $\partial D'$ are disjoint and essential in F. If $c(\partial D \cup \partial D') = c(\partial D)$ (say), then $\langle \partial D' \rangle \subset \langle \partial D \rangle$, so $\partial D'$ bounds a disk in W. Since it also bounds a disk in W', we conclude that the splitting (W, W') is reducible.

Let $\boldsymbol{D}$, $\boldsymbol{D}'$ be disjoint unions of disks in W, W' respectively such that

$$\partial\boldsymbol{D}\cap\partial\boldsymbol{D}'=\emptyset, \qquad c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')>c(\partial\boldsymbol{D}), c(\partial\boldsymbol{D}'),$$

and with $c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')$ maximal subject to these conditions. By the preceding remarks, we may assume that such $\boldsymbol{D}$, $\boldsymbol{D}'$ exist.

Let $N(\boldsymbol{D})$, $N(\boldsymbol{D}')$ be regular neighborhoods of $\boldsymbol{D}$, $\boldsymbol{D}'$ in W, W' respectively such that $N(\boldsymbol{D})\cap F$ and $N(\boldsymbol{D}')\cap F$ are disjoint regular neighborhoods of $\partial\boldsymbol{D}$ and $\partial\boldsymbol{D}'$ in F. Let $W_0=\overline{W-N(\boldsymbol{D})}$ and $W_0'=\overline{W'-N(\boldsymbol{D}')}$; W_0 and W_0' are disjoint unions of handlebodies. Let $Q=W_0\cup N(\boldsymbol{D}')$ and $Q'=W_0'\cup N(\boldsymbol{D})$. Then $\partial Q=\partial Q'=F_0$, say, where $F_0\cong\sigma(F;\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')$.

Assume, as one may, that M contains no incompressible surface of positive genus, and suppose that F_0 is not a disjoint union of 2-spheres. Then F_0 is compressible, say into Q. Let $F\times I$ be a collar of F in W', such that $N(\boldsymbol{D}')\cap F\times I=(N(\boldsymbol{D}')\cap F)\times I$, and extend $(F\cap\partial W_0)\times I$ to a collar $\partial W_0\times I$ of ∂W_0 in $\overline{M-W_0}$. Let $V=\partial W_0\times I\cup N(\boldsymbol{D}')$. Then $Q\cong W_0\cup V$, and V is obtained from a compression body $\hat{V}$, say, with $\partial_+\hat{V}=\partial W_0$, by removing some open 3-balls. Since F_0 is compressible in Q, Lemma 1.1 implies that there exists a disk D in $W_0\cup V$ with ∂D essential in $\partial(W_0\cup V)$, and a disjoint union of disks $\boldsymbol{E}'$ in V which is a complete disk system for $\hat{V}$, such that $D\cap W_0$ is a disk D_0 with $\partial D_0\cap\partial\boldsymbol{E}'=\emptyset$. Since ∂W_0-F consists of disks, isotopies of D_0 and $\boldsymbol{E}'$ will ensure that ∂D_0 and $\partial\boldsymbol{E}'$ lie in $F\cap\partial W_0$ (still keeping $\partial D_0\cap\partial\boldsymbol{E}'=\emptyset$). Let $\boldsymbol{E}=\boldsymbol{D}\cup D_0$. Then $\boldsymbol{E}$, $\boldsymbol{E}'$ are disjoint unions of disks in W, W' respectively such that $\partial\boldsymbol{E}\cap\partial\boldsymbol{E}'=\emptyset$.

Note that $c(\partial\boldsymbol{E}\cup\partial\boldsymbol{E}')>c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')$. For

$$\begin{aligned}\sigma(F;\partial\boldsymbol{E}\cup\partial\boldsymbol{E}')&=\sigma(\sigma(F;\partial\boldsymbol{D});\partial\boldsymbol{E}'\cup\partial D_0)\\&\cong\sigma(\partial_+\hat{V};\partial\boldsymbol{E}'\cup\partial D_0)\\&\cong\sigma(\partial_-\hat{V};\partial D)\quad\text{modulo 2-spheres}\end{aligned}$$

has complexity less than that of $\partial_-\hat{V}$ by hypothesis, and

$$\sigma(F;\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')\cong\partial_-\hat{V}\quad\text{modulo 2-spheres.}$$

Next, suppose $c(\partial\boldsymbol{E}\cup\partial\boldsymbol{E}')=c(\partial\boldsymbol{E})$. Then $\langle\partial\boldsymbol{E}'\rangle\subset\langle\partial\boldsymbol{E}\rangle$, so that each component of $\partial\boldsymbol{E}'$ bounds disks in both W and W'. This implies that (W, W') is reducible unless $c(\partial\boldsymbol{E}')=0$. Similarly, if $c(\partial\boldsymbol{E}\cup\partial\boldsymbol{E}')=c(\partial\boldsymbol{E}')$, we are done unless $c(\partial\boldsymbol{E})=0$. But since $\partial\boldsymbol{E}=\partial\boldsymbol{D}\cup\partial D_0$ contains an essential curve, $c(\partial\boldsymbol{E})>0$. Also, $c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')=c(\partial\boldsymbol{D}\cup\partial\boldsymbol{E}')$ (since, modulo 2-spheres, $\sigma(\partial_+\hat{V};\partial\boldsymbol{E}')\cong\partial_-\hat{V}\cong\sigma(\partial_+\hat{V};\partial\boldsymbol{D}')$), and so $c(\partial\boldsymbol{E}')=0$ would imply that $c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')=c(\partial\boldsymbol{D})$, contradicting our assumption on $\boldsymbol{D}$, $\boldsymbol{D}'$. Therefore we may assume that $c(\partial\boldsymbol{E}\cup\partial\boldsymbol{E}')>c(\partial\boldsymbol{E}), c(\partial\boldsymbol{E}')$. Since this contradicts our assumption that $c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')$ was maximal subject to $c(\partial\boldsymbol{D}\cup\partial\boldsymbol{D}')>c(\partial\boldsymbol{D})$, $c(\partial\boldsymbol{D}')$, we conclude that F_0 is a disjoint union of 2-spheres.

Since F is connected and $\partial\boldsymbol{D}\neq\emptyset\neq\partial\boldsymbol{D}'$, there exists a component S of F_0 such that each of the disjoint unions of open disks $S\cap\operatorname{int}W$, $S\cap\operatorname{int}W'$ is non-empty. Let α be a simple loop in S which separates $S\cap\operatorname{int}W$ from $S\cap\operatorname{int}W'$. Then clearly

α bounds disks in W and W'. Finally, since we may assume without loss of generality that each component of $\partial \boldsymbol{D} \cup \partial \boldsymbol{D}'$ is essential in F, α is essential in F. Hence the splitting (W, W') is reducible. $\square$

4. Roots in the fundamental group of a compression body

Let W be a handlebody and α a loop in ∂W. Let $[\alpha]$ denote the conjugacy class in $\pi_1(W)$ represented by α. (Note that every conjugacy class in $\pi_1(M)$ is represented by some loop in ∂W.) B. Maskit asked the following question: if α and β are loops in ∂W such that $[\alpha]=[\beta^n]$ and α is simple, can β be chosen to be simple? We shall show that this is indeed the case; this was also proved independently by P. Scott (unpublished). (In [15, Remark 6.2] Maskit credits us and Scott with a natural generalization to the case of several disjoint simple loops. However, this generalization does not seem to be straightforward.)

The following theorem shows that Maskit's question, and its extension to compression bodies W, has an affirmative answer. Again note that every conjugacy class in $\pi_1(W)$ is represented by a loop in $\partial_+ W$.

Theorem 4.1. *Let W be a compression body and let α be an essential simple loop in $\partial_+ W$ such that $[\alpha]=[\beta^n]$ in $\pi_1(W)$ for some loop β in $\partial_+ W$ and some integer n greater than* 1. *Then W contains a solid torus V as a boundary connected summand such that α lies in ∂V. Hence β can be chosen to be a simple loop in $\partial V \cap \partial_+ W$.*

Proof. We may assume that $[\alpha] \neq \{1\}$.

Let $F=\partial_+ W$ and let W' be the compression body $W(F; \alpha)$. Consider the 3-manifold $M = W \bigcup_F W'$. Note that $\pi_1(M) \cong \pi_1(W)/\langle\beta^n\rangle$. We claim that (any representative of) $[\beta]$ has order n in $\pi_1(M)$. In the case that W is a handlebody, then $\pi_1(W)$ is free and this is a known property of 1-relator groups [13]. In general, $\pi_1(W)$ is a free product of a free group and closed surface groups. Since closed surface groups are residually free [1, 7, 14] and since residual freeness is preserved under free product , $\pi_1(W)$ is residually free. The claim then follows easily from the corresponding statement for free groups. Thus $\pi_1(W)$ has torsion. Therefore either M is closed (and $\pi_1(M)$ is finite) or $\pi_2(M) \neq 0$. If M is closed, then W must be a solid torus, in which case the theorem is trivially true.

So we may assume that $\pi_2(M) \neq 0$. Then, by the sphere theorem, M contains an essential 2-sphere. By Lemma 1.1, M contains an essential 2-sphere S which meets F in a single circle and is disjoint from some complete disk system $\boldsymbol{D}'$ for W'. Since any two complete disk systems are related by 'handle-slides' [2, Appendix B], $\boldsymbol{D}'$ is a single disk with $\partial \boldsymbol{D}' = \alpha$. Let E and E' be the disks $S \cap W$, $S \cap W'$, respectively.

Since $\partial E' \in \langle\alpha\rangle$ (in $\pi_1(F)$), either $\partial E'$ is parallel to α, or $\partial E'$ bounds a punctured torus T_0 in F with $\alpha \subset T_0$. The first case is impossible since α would then bound a disk in W, contradicting $[\alpha] \neq \{1\}$ in $\pi_1(W)$. The second case implies that E

decomposes W as a boundary connected sum $W_0 \#_\partial V$, where ∂V is the torus $T_0 \cup E$. Since $[\alpha]=[\beta^n]$ in $\pi_1(W) \cong \pi_1(W_0) * \pi_1(V)$, it follows from standard properties of free products that α represents an nth power in $\pi_1(V)$. On the other hand, since α is simple, it represents a primitive element of $\pi_1(\partial V)$. Hence ∂V compresses in W, which implies that V is a solid torus. $\square$

References

[1] G. Baumslag, On generalized free products, Math. Z. 78 (1962) 423-438.
[2] F. Bonahon, Cobordism of automorphisms of surfaces, Ann. Sci. Ec. Norm. Sup. 16 (4) (1983) 237-270.
[3] F. Bonahon and J.-P. Otal, Scindements de Heegaard des espaces lenticulaires, Ann. Sci. Ec. Norm. Sup. 16 (4) (1983) 451-466.
[4] A.J. Casson and C.McA. Gordon, Reducing Heegaard splittings of 3-manifolds, Abstracts Amer. Math. Soc. 4 (2) (1983) 182.
[5] A.J. Casson and C.McA. Gordon, Manifolds with irreducible Heegaard spittings of arbitrarily high genus, to appear.
[6] M. Domergue and H. Short, Surfaces incompressibles dans les variétés obtenues par chirurgie longitudinale le long d'un noeud de S^3, preprint.
[7] K. Frederick, Hopfian property of a class of fundamental groups, Comm. Pure Appl. Math. 16 (1963) 1-8.
[8] W. Haken, Some results on surfaces in 3-manifolds, Studies in Modern Topology (Math. Assoc. Amer., distributed by: Prentice-Hall, 1968) 34-98.
[9] J. Hempel, 3-Manifolds, Ann. Math. Studies 86 (Princeton University Press, Princeton, NJ, 1976).
[10] W. Jaco, Lectures on Three-Manifold Topology, CBMS Regional Conference Series in Math. 43 (Amer. Math. Soc., Providence, RI, 1980).
[11] W. Jaco, Adding a 2-handle to 3-manifold: an application to Property R, Proc. Amer. Math. Soc. 92 (1984) 288-292.
[12] K. Johannson, On surfaces in one-relator 3-manifolds, preprint.
[13] A. Karrass, W. Magnus and D. Solitar, Elements of finite order in a group with a single defining relation, Comm. Pure Appl. Math. 13 (1960) 57-66.
[14] D.D. Long, Planar kernels in surface groups, Quart. J. Math. Oxford 35 (1984) 305-309.
[15] B. Maskit, Parabolic elements in Kleinian groups, Ann. Math. 117 (1983) 659-668.
[16] J. Milnor, A unique factorization theorem for 3-manifolds, Amer. J. Math. 84 (1962) 1-7.
[17] J.H. Przytycki, Incompressibility of surfaces after Dehn surgery, Michigan Math. J. 30 (1983) 289-308.
[18] M. Scharlemann, Outermost forks and a theorem of Jaco, Proc. Rochester Confer., 1982.
[19] F. Waldhausen, Some problems on 3-manifolds, Proc. Symp. Pure Math. 32 (1978) 313-322.